

Determinante

matrica tipa $n \times n$

Determinanta je broj pridružen svakoj kvadratnoj matrici. Determinantu matrice A obilježavamo sa $\det A$ ili $|A|$.

Preciznija definicija determinante je:

Determinanta je f-ja koja $n \times n$ realnih brojeva preslikava u realan broj.

Osobine determinante: (neke osobine determinanti)

1. Determinanta jedinične matrice je 1 ($\det I = 1$).
2. Ako dva reda (ili dvije kolone) međusobno zamjene mjesto znak determinante se mijenja.

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \quad \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

3. a) Determinanta se množi jednim brojem ako se tim brojem pomnože svi elementi jednog reda (ili, jedne kolone)

$$t \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} ta & tb \\ tc & td \end{vmatrix} \quad b) \begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

(linearnost za svaki red)

1. Izračunati: $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

$$a) \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{vmatrix} \stackrel{R_3}{=} 2 \cdot \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 2 \cdot 1 = 2$$

razvoj determinante po trećem redu

$$b) \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix} \stackrel{\text{razvoj determinante po prvom redu}}{=} 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = 1 \cdot 0 - 2 \cdot (-3) + 0 = 6$$

Mogli smo izračunati i na sljedeći način:

$$\begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix} \stackrel{\text{III}_k - \text{II}_k}{=} \begin{vmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & 1 \end{vmatrix} = (-2) \cdot \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = (-2) \cdot (-3) = 6$$

2. Izračunati:

$$a) \begin{vmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} \stackrel{\text{III}_k - \text{IV}_k}{=} \begin{vmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = (-1) \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -2$$

$$b) \begin{vmatrix} 4 & 1 & 0 & 3 \\ 2 & 0 & 1 & 1 \\ 4 & 1 & 0 & 1 \\ 0 & 1 & 0 & 3 \end{vmatrix} \stackrel{\text{I}_k - \text{IV}_k}{=} \begin{vmatrix} 4 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 4 & 1 & 0 & 1 \\ 0 & 1 & 0 & 3 \end{vmatrix} = 4 \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 3 \end{vmatrix} = 4 \cdot (-1) \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = (-4) \cdot 2 = -8$$

3. Izračunati:

$$a) \begin{vmatrix} 3 & -2 & 1 \\ 4 & -1 & 1 \\ 1 & 1 & 5 \end{vmatrix} \stackrel{R_2 - R_1}{=} \begin{vmatrix} -1 & -1 & 0 \\ 4 & -1 & 1 \\ 1 & 1 & 5 \end{vmatrix} \stackrel{\text{III}_k + \text{I}_k}{=} \begin{vmatrix} -1 & -1 & 0 \\ 4 & -1 & 1 \\ 0 & 0 & 5 \end{vmatrix} = 5 \begin{vmatrix} -1 & -1 \\ 4 & -1 \end{vmatrix} = 5 \cdot 5 = 25$$

$$b) \begin{vmatrix} 1 & 3 & 3 \\ 2 & -1 & 4 \\ 1 & 2 & 7 \end{vmatrix} \stackrel{\text{I}_k - \text{III}_k}{=} \begin{vmatrix} 0 & -1 & -4 \\ 2 & -1 & 4 \\ 1 & 2 & 7 \end{vmatrix} \stackrel{\text{II}_k + \text{I}_k}{=} \begin{vmatrix} 0 & -1 & -4 \\ 2 & 0 & 0 \\ 1 & 2 & 7 \end{vmatrix} = (-2) \cdot \begin{vmatrix} 1 & -4 \\ 2 & 7 \end{vmatrix} = (-2) \cdot 15 = -30$$

4. Izračunati:

$$a) \begin{vmatrix} 1 & 0 & 1 & 0 \\ 2 & 5 & 2 & 0 \\ 3 & 0 & 0 & 1 \\ 4 & 3 & 2 & 1 \end{vmatrix} \stackrel{R_2 - R_1}{=} \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 5 & 0 & 0 \\ 3 & 0 & -3 & 1 \\ 4 & 3 & -2 & 1 \end{vmatrix} \stackrel{\text{III}_k - \text{I}_k \cdot 3}{=} \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 3 & -2 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 5 & 0 & 0 \\ 0 & -3 & 1 \\ 3 & -2 & 1 \end{vmatrix} = 5 \cdot \begin{vmatrix} -3 & 1 \\ -2 & 1 \end{vmatrix} = 5 \cdot (-1) = -5$$

$$b) \begin{vmatrix} 0 & 0 & 1 & 2 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 5 \end{vmatrix}$$

$$c) \begin{vmatrix} 5 & 4 & 3 & 2 \\ 1 & 1 & 2 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \end{vmatrix}$$

riješenja:

b) 0 c) -1

5. Izračunati:

$$\begin{vmatrix} \sqrt{3} & 2\sqrt{2} & \sqrt{5} \\ 5\sqrt{3} & \sqrt{8} & 7\sqrt{5} \\ \sqrt{5} + 2\sqrt{3} & 4\sqrt{2} & \sqrt{3} + 2\sqrt{5} \end{vmatrix} \quad R_2: 36\sqrt{2}$$

6.) Dokazati da je $\begin{vmatrix} 1 & a & a^2+a^3 \\ 1 & a^2 & a^3+a \\ 1 & a^3 & a+a^2 \end{vmatrix} = 0$.

Rj. $\begin{vmatrix} 1 & a & a^2+a^3 \\ 1 & a^2 & a^3+a \\ 1 & a^3 & a+a^2 \end{vmatrix} = a \begin{vmatrix} 1 & 1 & a^2(1+a) \\ 1 & a & a(a^2+1) \\ 1 & a^2 & a(1+a) \end{vmatrix} = a \cdot a \cdot \begin{vmatrix} 1 & 1 & a(a+1) \\ 1 & a & a^2+1 \\ 1 & a^2 & a+1 \end{vmatrix} \xrightarrow{\substack{II_R - I_R \\ III_R - I_R}}$

$$= a^2 \begin{vmatrix} 1 & 1 & a(a+1) \\ 0 & a-1 & 1-a \\ 0 & a^2-1 & 1-a^2 \end{vmatrix} = a^2 \begin{vmatrix} a-1 & 1-a \\ (a+1)(a-1) & 1-a^2 \end{vmatrix} = a^2(a-1) \begin{vmatrix} 1 & 1-a \\ a+1 & (1-a)(1+a) \end{vmatrix}$$

$$= a^2(a-1)(1-a) \begin{vmatrix} 1 & 1 \\ a+1 & a+1 \end{vmatrix} = a^2(a-1)(1-a)(a+1) \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

što je i trebalo dobiti.

7.) Izračunati: $\begin{vmatrix} a & b & a & b \\ b & a & a & b \\ a & b & b & a \\ b & a & b & a \end{vmatrix}$

Rj. $\xrightarrow{N_k + (I_k + III_k + V_k)} \begin{vmatrix} a & b & a & 2a+2b \\ b & a & a & 2a+2b \\ a & b & b & 2a+2b \\ b & a & b & 2a+2b \end{vmatrix}$

$$= (2a+2b) \begin{vmatrix} a & b & a & 1 \\ b & a & a & 1 \\ a & b & b & 1 \\ b & a & b & 1 \end{vmatrix} \xrightarrow{\substack{II_R - I_R \\ III_R - I_R \\ IV_R - I_R}} (2a+2b) \begin{vmatrix} a & b & a & 1 \\ b-a & a-b & 0 & 0 \\ a & b & b & 1 \\ b-a & a-b & 0 & 0 \end{vmatrix} \xrightarrow{II_R - I_R} (2a+2b) \begin{vmatrix} a & b & a & 1 \\ b-a & a-b & 0 & 0 \\ a & b & b & 1 \\ b-a & a-b & 0 & 0 \end{vmatrix}$$

$$= (2a+2b) \begin{vmatrix} a & b & a & 1 \\ b-a & a-b & 0 & 0 \\ 0 & 0 & b-a & 1 \\ b-a & a-b & 0 & 0 \end{vmatrix} = (2a+2b) \begin{vmatrix} a & b & a \\ b-a & a-b & 0 \\ b-a & a-b & 0 \end{vmatrix} = -a(2a+2b) \begin{vmatrix} b-a & a-b \\ ba & a-b \end{vmatrix}$$

$$= -a(2a+2b)(b-a)(a-b) \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

8.) Rastaviti na faktore polinom:

a) $\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$ b) $\begin{vmatrix} a & b & a+b \\ b & a+b & a \\ a+b & a & b \end{vmatrix}$ c) $\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 \\ b^2 & (b+1)^2 & (b+2)^2 \\ c^2 & (c+1)^2 & (c+2)^2 \end{vmatrix}$

Riješiti jednačinu $\begin{vmatrix} 3x-5 & -5-2x & x+1 \\ 2x-4 & -2-2x & x-1 \\ 3x-8 & 2-3x & 2x-5 \end{vmatrix} = 0$

Rj. $\begin{vmatrix} 3x-5 & -5-2x & x+1 \\ 2x-4 & -2-2x & x-1 \\ 3x-8 & 2-3x & 2x-5 \end{vmatrix} = (-1) \begin{vmatrix} 3x-5 & 2x+5 & x+1 \\ 2x-4 & 2x+2 & x-1 \\ 3x-8 & 3x-2 & 2x-5 \end{vmatrix} \xrightarrow{III_V - II_V}$

$$\begin{vmatrix} 3x-5 & 2x+5 & x+1 \\ 2x-4 & 2x+2 & x-1 \\ x-4 & x-4 & x-4 \end{vmatrix} = (-1)(x-4) \begin{vmatrix} 3x-5 & 2x+5 & x+1 \\ 2x-4 & 2x+2 & x-1 \\ 1 & 1 & 1 \end{vmatrix} \xrightarrow{\substack{I_k - III_k \\ II_k - III_k}}$$

$$= (-1)(x-4) \begin{vmatrix} 2x-6 & x+4 & x+1 \\ x-3 & x+3 & x-1 \\ 0 & 0 & 1 \end{vmatrix} = (-1)(x-4) \begin{vmatrix} 2x-6 & x+4 \\ x-3 & x+3 \end{vmatrix} \xrightarrow{I_V - II_V}$$

$$= (-1)(x-4) \begin{vmatrix} x-3 & 1 \\ x-3 & x+3 \end{vmatrix} = (-1)(x-4)(x-3) \begin{vmatrix} 1 & 1 \\ 1 & x+3 \end{vmatrix} = (-1)(x-4)(x-3)(x+2)$$

$(-1)(x-4)(x-3)(x+2) = 0$ Rješenja jednačine su $x=4$ ili $x=3$ ili $x=-2$.

Riješiti jednačinu: $\begin{vmatrix} x-3 & x+2 & x-1 \\ x+2 & x-4 & x \\ x-1 & x+4 & x-5 \end{vmatrix} = 0$

Rj. $\begin{vmatrix} x-3 & x+2 & x-1 \\ x+2 & x-4 & x \\ x-1 & x+4 & x-5 \end{vmatrix} \xrightarrow{I_k + III_k + V_k} \begin{vmatrix} 3x-2 & x+2 & x-1 \\ 3x-2 & x-4 & x \\ 3x-2 & x+4 & x-5 \end{vmatrix} = (3x-2) \begin{vmatrix} 1 & x+2 & x-1 \\ 1 & x-4 & x \\ 1 & x+4 & x-5 \end{vmatrix}$

$$\xrightarrow{\substack{I_k - II_k \\ II_k - III_k}} (3x-2) \begin{vmatrix} 0 & 6 & -1 \\ 1 & x-4 & x \\ 0 & 8 & -5 \end{vmatrix} = -(3x-2) \begin{vmatrix} 6 & -1 \\ 8 & -5 \end{vmatrix} = -(3x-2)(-30+8) =$$

$$= 22(3x-2) \quad 22(3x-2) = 0 \quad 3x-2 \text{ je rješenje jednačine}$$

$$3x-2 = 0 \quad x = \frac{2}{3}$$

Matematičkom indukcijom dokazati:

$$\begin{vmatrix} 1+x^2 & x & 0 & \dots & 0 & 0 \\ x & 1+x^2 & x & \dots & 0 & 0 \\ 0 & x & 1+x^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1+x^2 & x \\ 0 & 0 & 0 & \dots & x & 1+x^2 \end{vmatrix} = 1+x^2+x^4+\dots+x^{2n}$$

(determinanta ima n vrsta i n kolona).

R) BAZA INDUKCIJE

Pokažimo da je tvrdnja tačna za broj 2

$$\begin{vmatrix} 1+x^2 & x \\ x & 1+x^2 \end{vmatrix} = (1+x^2)^2 - x^2 = 1+2x^2+x^4-x^2 = 1+x^2+x^4$$

Jednakost je tačna za broj 2.

KORAK INDUKCIJE

Pretpostavimo da je jednakost tačna za determinantu koja ima k vrsta i k kolona

$$\begin{vmatrix} 1+x^2 & x & \dots & 0 & 0 \\ x & 1+x^2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1+x^2 & x \\ 0 & 0 & \dots & x & 1+x^2 \end{vmatrix} = 1+x^2+x^4+\dots+x^{2k}$$

gdje k uzima brojeve od 1 do n. Na osnovu ove pretpostavke dokažimo da je jednakost tačna za determinantu koja ima n+1 vrsta i n+1 kolona tačnije dokažimo da

$$\begin{vmatrix} 1+x^2 & x & \dots & 0 & 0 \\ x & 1+x^2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1+x^2 & x \\ 0 & 0 & \dots & x & 1+x^2 \end{vmatrix} = 1+x^2+x^4+\dots+x^{2n}+x^{2n+2}$$

Polazimo od determinante koja ima (n+1)-vrsta i (n+1)-kolona:

$$\begin{vmatrix} 1+x^2 & x & \dots & 0 & 0 \\ x & 1+x^2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1+x^2 & x \\ 0 & 0 & \dots & x & 1+x^2 \end{vmatrix} \begin{matrix} \text{razuj} \\ \text{po prvaj} \\ \text{koloni} \end{matrix} (1+x^2) \begin{vmatrix} 1+x^2 & x & \dots & 0 & 0 \\ x & 1+x^2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1+x^2 & x \\ 0 & 0 & \dots & x & 1+x^2 \end{vmatrix} - x \begin{vmatrix} x & 0 & \dots & 0 & 0 \\ x & 1+x^2 & \dots & 0 & 0 \\ 0 & x & 1+x^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1+x^2 & x \\ 0 & 0 & \dots & x & 1+x^2 \end{vmatrix} =$$

na osnovu pretpostavke (ovu determinantu mogu razviti po prvaj vrsti i ostade ni determinanta iz pretpostavke koja ima n-1 vrsta i n-1 kolona) (ovu determinantu mogu razviti po prvaj vrsti i ostade ni determinanta iz pretpostavke koja ima n-1 vrsta i n-1 kolona) što je i trebalo dobiti

ZAKLJUČAK

Jednakost je tačna za sve prirodne brojeve

Izračunati

$$\begin{vmatrix} 1 & a & 3 & 2 \\ 2 & 2 & -2 & 1 \\ 3 & 3 & -5 & 1 \\ 4 & 4 & -7 & 5 \end{vmatrix}$$

R.)

$$\begin{vmatrix} 1 & a & 3 & 2 \\ 2 & 2 & -2 & 1 \\ 3 & 3 & -5 & 1 \\ 4 & 4 & -7 & 5 \end{vmatrix} \begin{matrix} I_k + III_k \\ II_k + III_k \\ III_k + IV_k \cdot 2 \end{matrix} \begin{vmatrix} 4 & a+3 & 7 & 2 \\ 0 & 0 & 0 & 1 \\ -2 & -2 & -3 & 1 \\ -3 & -3 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 4 & a+3 & 7 \\ -2 & -2 & -3 \\ -3 & -3 & 3 \end{vmatrix} \begin{matrix} I_k + III_k \\ II_k + III_k \end{matrix}$$

$$= \begin{vmatrix} 11 & a+10 & 7 \\ -5 & -5 & -3 \\ 0 & 0 & 3 \end{vmatrix} = 3 \begin{vmatrix} 11 & a+10 \\ -5 & -5 \end{vmatrix} = 3 \cdot (-5) \begin{vmatrix} 11 & a+10 \\ 1 & 1 \end{vmatrix} = -15(11-a-10)$$

$$= -15(-a+1) = 15a - 15$$

#) Matematičkom indukcijom dokazati:

$$\begin{vmatrix} 1 & n & n & \dots & n & n \\ n & 2 & n & \dots & n & n \\ n & n & 3 & \dots & n & n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n & n & n & \dots & n-1 & n \\ n & n & n & \dots & n & n \end{vmatrix} = (-1)^{n-1} \cdot n!$$

Rj. BAZA INDUKCIJE

Pokažimo da je tvrdnja tačna za broj 2.

$$\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = 2-4 = -2 = (-1)^{2-1} \cdot 2! \quad \text{Jednakost je tačna za broj 2.}$$

KORAK INDUKCIJE

Pretpostavimo da je jednakost

$$\begin{vmatrix} 1 & k & k & \dots & k & k \\ k & 2 & k & \dots & k & k \\ k & k & 3 & \dots & k & k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k & k & k & \dots & k-1 & k \\ k & k & k & \dots & k & k \end{vmatrix} = (-1)^{k-1} \cdot k!$$

tačna za sve brojeve od

1 do n ($k=1, 2, \dots, n$).

Uz pomoć ove pretpostavke

dokažimo da je jednakost tačna za broj n+1 tj. dokažimo

$$\begin{vmatrix} 1 & n+1 & \dots & n+1 & n+1 \\ n+1 & 2 & \dots & n+1 & n+1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ n+1 & n+1 & \dots & n & n+1 \\ n+1 & n+1 & \dots & n+1 & n+1 \end{vmatrix} = (-1)^n \cdot (n+1)!$$

ZAKLJUČAK
Jednakost je tačna za sve prirodne brojeve

$$\begin{aligned} & \begin{vmatrix} 1 & n+1 & \dots & n+1 & n+1 \\ n+1 & 2 & \dots & n+1 & n+1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ n+1 & n+1 & \dots & n & n+1 \\ n+1 & n+1 & \dots & n+1 & n+1 \end{vmatrix} \xrightarrow{|_k - (N+1)_k} \begin{vmatrix} -n & n+1 & \dots & n+1 & n+1 \\ 0 & 2 & \dots & n+1 & n+1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & n+1 & \dots & n & n+1 \\ 0 & n+1 & \dots & n+1 & n+1 \end{vmatrix} = \\ & = (-n) \begin{vmatrix} 2 & n+1 & \dots & n+1 & n+1 \\ n+1 & 3 & \dots & n+1 & n+1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ n+1 & n+1 & \dots & n & n+1 \\ n+1 & n+1 & \dots & n+1 & n+1 \end{vmatrix} = (-n)(n+1) \begin{vmatrix} 2 & n+1 & \dots & n+1 & 1 \\ n+1 & 3 & \dots & n+1 & 1 \\ \vdots & \vdots & \ddots & \vdots & 1 \\ n+1 & n+1 & \dots & n & 1 \\ n+1 & n+1 & \dots & n+1 & 1 \end{vmatrix} \xrightarrow{\substack{|_k - N_k \\ |_{k-1} - N_{k-1} \\ \vdots \\ |(N-1)_k - N_k}} \\ & = (-1) \cdot n(n+1) \begin{vmatrix} 1 & n & \dots & n & 1 \\ n & 2 & \dots & n & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ n & n & \dots & n-1 & 1 \\ n & n & \dots & n & 1 \end{vmatrix} \xrightarrow{\substack{\text{na osnovu} \\ \text{pretpostavke}}} (-1)(n+1)(-1)^{n-1} \cdot n! \\ & = (-1)^n (n+1)! \end{aligned}$$